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On Banach Spaces with Bases

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We show that a Banach space X has a basis provided there are bounded linear finite rank operators $R_n: X \to X$ such that $\lim_n R_n x = x$ for all $x \in X$, $R_m R_n = R_{\min(m,n)}$ if $m \neq n$, and $R_n - R_{n-1}$ factors uniformly through $l_p^{m_n}$'s for some p. As an application we obtain conditions on a subset $\Lambda \subset \mathbf{Z}$ such that $C_\Lambda = \text{closed span}\{z^k : k \in \Lambda\} \subset C(\mathbf{T})$ and $L_\Lambda = \text{closed span}\{z^k : k \in \Lambda\} \subset L_1(\mathbf{T})$ have bases.

1. Introduction

Let X be a given separable (real or complex) Banach space. We want to give a sufficient condition for X to have a basis, which can be easily verified in concrete situations where explicit constructions of bases are extremely complicated or even unknown.

Fix some p with $1 \leqslant p \leqslant \infty$. We say that a sequence of linear operators $U_n: X \to X$ factors uniformly through l_p^m's with respect to λ, if there are suitable integers m_n and linear operators

$$T_n: X \to l_p^{m_n}, \qquad S_n: l_p^{m_n} \to X$$

with

$$U_n = S_n T_n \qquad \text{and} \qquad \sup_n \|T_n\|, \sup_n \|S_n\| \leqslant \lambda \tag{1.1}$$

A sequence of bounded linear operators $R_n: X \to X$ of finite rank is called commuting approximating sequence (c.a.s.) if $\lim_n R_n x = x$ for all $x \in X$ and

$$R_n R_m = R_{\min(n,m)} \qquad \text{whenever} \quad n \neq m. \tag{1.2}$$

If there exists such a sequence $\{R_n\}_{n=1}^\infty$ then X is said to have the commuting bounded approximation property (CBAP). If (1.2) holds in addition even for all n, m with $n = m$ then X is said to have a finite dimensional Schauder decomposition (FDD). It is known that there are Banach spaces with CBAP which do not have FDD, [2], [11].

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At first we want to strengthen the main result of [7].

THEOREM I. *Let X have a c.a.s. $\{R_n\}_{n=1}^{\infty}$ such that $R_n - R_{n-1}$ factors uniformly through l_p^m's for some $1 \leqslant p \leqslant \infty$. Then X has a basis.*

We postpone the proof of Theorem I to Section 3. Here we give some applications. At first some remarks. Notice that, conversely, every space X with basis admits a sequence R_n satisfying the assumptions of Theorem I (take the basis projections). In Theorem I we cannot drop the assumption (1.2) on R_n: Consider a Banach space X with linear bounded finite rank operators $R_n: X \to X$ such that $R_n \to id$ pointwise (i.e. X has the bounded approximation property-BAP). It follows from [9] that in this case we always find R_n's with $\dim(R_n - R_{n-1})\, X = 1$ in addition, that is, $R_n - R_{n-1}$ factors uniformly through l_p^m's. There are however examples of separable Banach spaces with BAP but without bases, [12].

Now we turn to $\mathbf{T} = \{z \in \mathbf{C}: |z| = 1\}$. Consider a subset $\Lambda \subset \mathbf{Z}$ and put

$$C_\Lambda = C(\mathbf{T})\text{-closure of span}\{z^k: k \in \Lambda\}$$

$$L_\Lambda = L_1(\mathbf{T})\text{-closure of span}\{z^k: k \in \Lambda\}$$

It is easy to see that, for every Λ, these spaces have the CBAP. It is unknown however if they always have bases, [1], [12]. We want to give partial answers to this question.

Let $d(\cdot, \cdot)$ be the Banach–Mazur distance between two Banach spaces. In the following consider $a, b > 0$ and a sequence of integers n_k with

$$n_0 = 0 < n_1 < n_2 \cdots \text{ and } a \cdot n_k \leqslant n_{k+1} - n_k \leqslant b \cdot n_k \qquad \text{for all} \quad k \tag{1.3}$$

Put $n_{-k} = -n_k$ for all $k = 1, 2, \ldots$. (As an example consider $n_k = 2^k$.)

For a function $f: \mathbf{T} \to \mathbf{C}$ let $\hat{f}(j)$ be its j'th Fourier coefficient. Then we have

THEOREM II. *Put $\Lambda_k = \Lambda \cap [n_k, n_{k+2}]$, $k = 0, \pm 1, \pm 2, \ldots$. Assume that there is a number $\lambda > 1$ and, for each k, there exists a finite-dimensional subspace $X_k \subset C(\mathbf{T})$ with $C_{\Lambda_k} \subset X_k$ and $d(X_k, l_\infty^{\dim X_k}) \leqslant \lambda$ (resp. $X_k \subset L_1(\mathbf{T})$ with $L_{\Lambda_k} \subset X_k$ and $d(X_k, l_1^{\dim X_k}) \leqslant \lambda$), satisfying the following: Whenever $f \in X_k$ and $n_k \leqslant j \leqslant n_{k+2}$ then $\hat{f}(j) = 0$ or $j \in \Lambda_k$. Then C_Λ (resp. L_λ) has a basis.*

Theorem II and the following assertions will be proven in Section 4. By a coset of $\mathbf{Z}$ we mean $\varnothing$ or a set of the form $m\mathbf{Z} + r$, where m and r are integers.

Corollary. *Assume that there are a positive integer q and cosets $Z_{k,j}$ of $\mathbf{Z}$, $j=1, ..., q$, with*

$$\Lambda \cap [n_k, n_{k+2}] = \bigcup_{j=1}^{q} (Z_{k,j} \cap [n_k, n_{k+2}]) \qquad \text{for} \quad k=0, \pm 1, \pm 2, ...$$

Then C_Λ and L_Λ have bases.

Theorem III. *Assume that there are a positive integer q and cosets $Z_{k,j}$ of $\mathbf{Z}$, $j=1, ..., q$, such that, with $\Omega_k = \bigcup_{j=1}^{q} Z_{k,j}$,*

$$\Lambda \cap [n_k, n_{k+1}] = \Omega_k \cap [n_k, n_{k+1}],$$

$$\Omega_k \cap \Omega_{k+1} \cap [n_k, n_{k+1}] = \varnothing \quad \text{for all } k=0, 1, ...,$$

and $\Omega_{k-1} \cap \Omega_k \cap [n_k, n_{k+1}] = \varnothing$ for all $k=-1, -2, ...$

Then C_Λ and L_Λ have bases.

2. Banach Spaces with a C.A.S. $\{R_n\}_{n=1}^{\infty}$ such that $R_n - R_{n-1}$ Factors Uniformly through l_p^m's

Fix a c.a.s. $\{R_n\}_{n=1}^{\infty}$ on X where $R_n - R_{n-1}$ factors uniformly through l_p^m's with respect to λ. Throughout this section we consider the linear operators $T_n: X \to l_p^{m_n}$ and $S_n: l_p^{m_n} \to X$ with

$$S_n T_n = R_n - R_{n-1} \qquad \text{and} \qquad \sup_n \|S_n\|, \sup_n \|T_n\| \leqslant \lambda. \tag{2.1}$$

(Put $R_0 = 0$.) For simplicity, take λ so large that $\sup_n \|R_n\| \leqslant \lambda$.

If $\dim X = \infty$ we want to assume that

$$\text{either } p=2 \text{ or } \sup_n d(T_n X, l_2^{\dim T_n X}) = \infty. \tag{2.2}$$

This is no restriction. Indeed, if $\sup_n d(T_n X, l_2^{\dim T_n X}) < \infty$ then we consider 2 instead of p, $T_n X$ instead of $l_p^{m_n}$ and $S_{n|T_n X}$ instead of S_n. Furthermore we can assume that

$$S_n l_p^{m_n} = (R_n - R_{n-1}) X \qquad \text{for all} \quad n. \tag{2.3}$$

Indeed, otherwise define

$$\tilde{T}_n: X \to (l_p^{m_{n-1}} \oplus l_p^{m_n} \oplus l_p^{m_{n+1}})_{(p)} =: E_n$$

by $\tilde{T}_n x = (T_{n-1}x, T_n x, T_{n+1}x)$. Put $\tilde{S}_n(a, b, c) = (R_n - R_{n-1})(S_{n-1}a + S_n b + S_{n+1}c)$, $(a, b, c) \in E_n$. Take $\tilde{T}_n$, $\tilde{S}_n$ instead of T_n, S_n, resp.

Let Fix $R_n = \{x\colon R_n x = x\}$ and define

$$Z = \left(\sum \oplus (R_n - R_{n-1})\, X\right)_{(p)}, \tag{2.4}$$

$$Y = \text{closed span}\left\{\{(R_n - R_{n-1})x\}_{n=1}^{\infty} : x \in \bigcup_n \text{Fix } R_n\right\}$$

(closure as subspace of Z)

and $i\colon \bigcup_n \text{Fix } R_n \to Y$ by $i(x) = \{(R_n - R_{n-1})x\}_{n=1}^{\infty}$, $x \in \bigcup_n \text{Fix } R_n$. Note that i is injective. The definition of R_m can be extended to Y by putting

$$R_m\{(R_n - R_{n-1})x\}_{n=1}^{\infty} = \{(R_n - R_{n-1})R_m x\}_{n=1}^{\infty}. \tag{2.5}$$

We obtain $i \circ R_m = R_m \circ i$. (The fact that we use the same notation for the operators R_m on X and on Y will not create any confusion lateron.)

LEMMA 2.1. (a) $\sup_m d((R_m - R_{m-1})\, Y, (R_m - R_{m-1})\, X) < \infty$.

(b) *There is a subspace* $Y \subset l_p$ *and a projection* $Q\colon l_p \to \tilde{Y}$ *such that* $d(Y, \tilde{Y}) < \infty$. $d(Y, \tilde{Y})$ *and* $\|Q\|$ *depend only on* λ.

(c) *If* $\dim Y = \infty$ *then* $d(Y, l_p)$ *is finite and depends only on* λ.

Proof. (a) We have

$$(R_m - R_{m-1})(R_n - R_{n-1}) = \begin{cases} R_{m-1}(id - R_{m-1}), & n = m-1 \\ (R_m - R_{m-1})^2, & n = m \\ R_m(id - R_m), & n = m+1 \\ 0, & \text{otherwise} \end{cases}$$

Hence

$$\begin{aligned}(R_m - R_{m-1})&\{(R_n - R_{n-1})x\}_{n=1}^{\infty} \\ &= (0, ..., 0, R_{m-1}(id - R_{m-1})x, (R_m - R_{m-1})^2 x, R_m(id - R_m)x, 0, 0, ...)\end{aligned}$$

This implies

$$\begin{aligned}\tfrac{1}{3}\, \|(R_m - R_{m-1})x\| &\leqslant \|(R_m - R_{m-1})\{(R_n - R_{n-1})x\}_{n=1}^{\infty}\| \\ &\leqslant 6\lambda\, \|(R_m - R_{m-1})x\|\end{aligned}$$

and hence (a).

(b) Put $W = (\sum \oplus l_p^{m_n})_{(p)}$, m_n as in (2.1). Hence $W \cong l_p$. Define $S\colon W \to Z$ by $S\{w_n\}_{n=1}^{\infty} = \{S_n w_n\}_{n=1}^{\infty}$. Clearly, S is well defined by (2.3) and $\|S\|$ depends only on λ. Moreover, for the elements $\{x_n\}_{n=1}^{\infty} \in Z$ which are eventually zero put

$$T\{x_n\}_{n=1}^{\infty} = \left\{T_n(R_{n+1} - R_{n-2})\left(\sum_{k=1}^{\infty} x_k\right)\right\}_{n=1}^{\infty}.$$

Then $T\{x_n\}_{n=1}^{\infty} \in W$. Indeed, by (1.2),

$$\|T\{x_n\}_{n=1}^{\infty}\| \leqslant \sum_{j=-2}^{2} \left(\sum_n \|T_n(R_{n+1} - R_{n-2})\, x_{n+j}\|^p\right)^{1/p} \leqslant 10\lambda^2 \,\|\{x_n\}_{n=1}^{\infty}\|.$$

So T can be extended to a bounded operator $T\colon Z \to W$. For $\{w_n\}_{n=1}^{\infty} \in W$ we obtain, by (2.1), (1.2),

$$\begin{aligned} TSTS\{w_n\}_{n=1}^{\infty} &= T\left\{S_n T_n(R_{n+1} - R_{n-2})\left(\sum_k S_k w_k\right)\right\}_{n=1}^{\infty} \\ &= T\left\{(R_n - R_{n-1})\left(\sum_k S_k w_k\right)\right\}_{n=1}^{\infty} \\ &= \left\{T_n(R_{n+1} - R_{n-2})\left(\sum_k S_k w_k\right)\right\}_{n=1}^{\infty}. \end{aligned}$$

(We have $\sum_n (R_n - R_{n-1}) = id$ on $\bigcup_n \operatorname{Fix} R_n$.) This implies $(TS)^2 = TS$. Hence TS is a bounded projection on $W \cong l_p$ and $S|_{TSW}$ is an isomorphism between TSW and Y. Since $\|TS\|$ and $\|S\|$ depend only on λ we obtain (b).

(c) is a consequence of (b). ∎

The preceding Lemma also includes the case that $\dim X < \infty$. Here there is an integer N with $R_N = R_{N+1} = \cdots = id$.

Lemma 2.2. *Let X be finite dimensional. Then there is an integer M, a subspace $F \subset l_p^M$ and a projection $Q\colon l_p^M \to F$ such that $X \oplus F$ has a basis. The corresponding basis constant as well as $\|Q\|$ depend only on λ.*

Proof. Consider Y of (2.4) which is now finite dimensional. Fix N such that $R_{N+j} = id$, $j = 0, 1, 2, \ldots$. Moreover, put $R_{-j} = 0$, $j = 0, 1, 2, \ldots$. Let $G = (\underbrace{Y \oplus \cdots \oplus Y}_{N\text{-times}})_{(p)}$ and define $P_m\colon X \oplus G \to X \oplus G$ by

$$\begin{aligned} P_m(x, \{y_k\}_{k=1}^{\infty}) = (&R_m x + i^{-1}(R_m - R_{m-1})\, y_m, (y_1, \ldots, y_{m-1}, (id - R_m)\, y_m \\ &+ i(R_{m+1} - R_m)\, x, 0, \ldots, 0)). \end{aligned}$$

Using (1.2), (2.4) we see that $\|i(R_{m+1} - R_m)\, x\|$ is bounded by a constant which depends only on λ and $\|x\|$. Similarly, (1.2) and (2.5) yield that

$\|i^{-1}(R_m - R_{m-1})\, y_m\|$ is bounded by a constant which depends only on λ and $\|y_m\|$ (see Lemma 2.1.(a)). Hence $\|P_m\|$ is bounded by a constant which depends only on λ. We obtain, by definition of P_m, $P_k P_m = P_m$ if $k > m$ and $P_j P_m = P_j$ if $j < m-1$. Moreover

$$\begin{aligned} P_m^2(x, \{y_k\}_{k=1}^{\infty}) = (R_m^2\, x + (R_m^2 - R_{m-1})\, i^{-1} y_m + (R_m - R_m^2)\, i^{-1} y_m \\ + (R_m - R_m^2)\, x, (y_1, ..., y_{m-1}, (id - R_m)^2\, y_m \\ + (R_{m+1} - R_m - (R_m - R_m^2))\, ix + (R_m - R_m^2)\, y_m \\ + (R_m - R_m^2)\, ix, 0, ..., 0)) = P_m(x, \{y_k\}_{k=1}^{\infty}). \end{aligned}$$

Hence P_m is a projection and $\{P_{2m}\}_{m=1}^{\infty}$ defines an FDD on $X \oplus G$. Moreover, $P_{N+1} = P_{N+2} = \cdots = id$. We claim, $P_{2m} - P_{2m-2}$ factors uniformly through l_p^k's and the factorization constant depends only on λ. Indeed, we have

$$\begin{aligned} &(P_{2m} - P_{2m-2})(x, \{y_k\}_{k=1}^{\infty}) \\ &\quad = ((R_{2m} - R_{2m-2})x + (R_{2m} - R_{2m-1})\, i^{-1} y_{2m} \\ &\qquad - (R_{2m-2} - R_{2m-3})\, i^{-1} y_{2m-2}, (\underbrace{0, ..., 0}_{2m-3}, R_{2m-2} y_{2m-2} \\ &\qquad - (R_{2m-1} - R_{2m-2})\, ix, y_{2m-1}, (id - R_{2m})\, y_{2m} \\ &\qquad + (R_{2m+1} - R_{2m})\, ix, 0, ..., 0)). \end{aligned}$$

Y is complemented in some l_p^k by Lemma 2.1. and $R_{2m} - R_{2m-2} = (R_{2m} - R_{2m-1}) - (R_{2m-1} - R_{2m-2})$ factors uniformly through l_p^m's too. Hence this proves the claim. Since $P_{2m} - P_{2m-2}$ is a projection we conclude that there are an $l_p^{k_m}$-space and subspaces A_m, B_m with $l_p^{k_m} = A_m \oplus B_m$, such that $(P_{2m} - P_{2m-2})(X \oplus G)$ is isomorphic to A_m and the isomorphism constant as well as the norm of the projection onto A_m along B_m depends only on λ. Finally, put $F = G \oplus (\sum \oplus B_m)_{(p)}$. Then $X \oplus F$ has an FDD whose summands are l_p^k-spaces and F is complemented in some l_p^M. Hence $X \oplus F$ has a basis. All constants involved depend only on λ. ∎

LEMMA 2.3. *Assume* $\sup_m \dim(R_m - R_{m-1})\, X = \infty$. *Then there is a subsequence* $\{\hat{R}_n\}_{n=1}^{\infty}$ *of* $\{R_m\}_{m=1}^{\infty}$ *and a constant* $c > 0$ *satisfying the following*:

(i) $\hat{R}_n - \hat{R}_{n-1}$ *factors uniformly through* l_p^m*'s, too.*

(ii) *For each positive integer* N *there is* n, *a subspace* $E_N \subset (\hat{R}_n - \hat{R}_{n-1})\, X$ *and a projection* $Q_N: X \to E_N$ *such that* $d(E_N, l_p^N) \leqslant c$, $\|Q_N\| \leqslant c$ *and*

$$\hat{R}_{n+m} Q_N = Q_N \hat{R}_{n+m} = Q_N, \quad m = 0, 1, 2, ..., \hat{R}_j Q_N = Q_N \hat{R}_j = 0 \qquad \textit{if} \quad j < n.$$

Proof. We can assume $\sup_n \dim(\mathrm{Fix}\, R_n \cap \ker R_{n-1}) = \infty$. Otherwise take $\{R_{3n}\}_{n=1}^{\infty}$ or $\{R_{3n+1}\}_{n=1}^{\infty}$ or $\{R_{3n+2}\}_{n=1}^{\infty}$ instead of $\{R_n\}_{n=1}^{\infty}$. (Note, $(R_n - R_{n-1})\, X \subset \mathrm{Fix}\, R_{n+1} \cap \ker R_{n-2}$).

The Case $2 \leqslant p < \infty$. In view of (2.2), by [10] Theorem B, for each N, there are n, a 2-isomorph $\tilde{E}_N \subset T_n(\mathrm{Fix}\, R_n \cap \ker R_{n-1})$ of l_p^N and a projection $\tilde{Q}_N : l_p^{m_n} \to \tilde{E}_N$ with $\|\tilde{Q}_N\| \leqslant 2$. S_n is an isomorphism on $T_n(\mathrm{Fix}\, R_n \cap \ker R_{n-1})$ (by (2.1)). Put $E_N = S_n \tilde{E}_N$ and $Q_N = S_n \tilde{Q}_N T_n(R_n - R_{n-1})$. Q_N is a projection onto $E_N \subset \mathrm{Fix}\, R_n \cap \ker R_{n-1}$. Hence

$$R_j Q_N = Q_N R_j = 0 \quad \text{if } j < n-1 \quad \text{and} \quad R_k Q_N = Q_N R_k = Q_N \quad \text{if } k > n.$$

By skipping $n-1$, n and using induction on N we obtain a sequence $\{\hat{R}_k\}_{k=1}^{\infty}$ as claimed.

The Case $1 < p \leqslant 2$. Here we consider $\hat{X} = \overline{\bigcup R_n^* X^*} \subset X^*$. Note that, as a consequence of (1.2), $R_1^* X^* \subset R_2^* X^* \subset \cdots$. Since $R_n^* - R_{n-1}^*$ factors uniformly through l_q^m's where $p^{-1} + q^{-1} = 1$ we proceed exactly as before to obtain a subsequence $\{\hat{R}_n^*\}_{n=1}^{\infty}$ of $\{R_n^*\}_{n=1}^{\infty}$, subspaces $E_N^* \subset (\hat{R}_n^* - \hat{R}_{n-1}^*)\, X^*$ and projections $Q_N^* \to E_N^*$ with all the properties claimed in the assertion of Lemma 2.3 for Q_N^* instead of Q_N, $\hat{R}_n^*$ instead of $\hat{R}_n$ and q instead of p. Regard X as subspace of X^{**}. Put $Q_N = Q_{N|X}^{**}$ and $E_N = Q_N X$. We have

$$Q_N^{**}(\hat{R}_n - \hat{R}_{n-1})^{**} = (\hat{R}_n - \hat{R}_{n-1})^{**}\, Q_N^{**} = Q_N^{**}$$

and

$$(\hat{R}_n - \hat{R}_{n-1})^{**}\, X^{**} = (\hat{R}_n - \hat{R}_{n-1})\, X.$$

Hence $E_N \subset (\hat{R}_n - \hat{R}_{n-1})\, X$. Thus Lemma 2.3 follows in this case.

The Case $p = \infty$. Here take a free ultrafilter $\mathscr{U}$ on the positive integers. Let U be the ultraproduct of $(R_n - R_{n-1})\, X$, $n = 1, 2, \ldots$, and V the ultraproduct of $l_\infty^{m_n}$, $n = 1, 2, \ldots$, with respect to $\mathscr{U}$. Define $T: U \to V$ by

$$T\{(R_n - R_{n-1})\, x_n\}_{n=1}^{\infty} = \{T_n(R_n - R_{n-1})\, x_n\}_{n=1}^{\infty}$$

and $S: V \to U$ by $S\{v_n\}_{n=1}^{\infty} = \{S_n v_n\}_{n=1}^{\infty}$. V is an L_1-predual space, hence V^{**} is a $C(K)$-space, [4]. S^{**} is not weakly compact. Indeed, consider

$$\tilde{U} = \{\{T_n(R_n - R_{n-1})\, x_n\}_{n=1}^{\infty} : x_n \in \mathrm{Fix}\, R_n \cap \ker R_{n-1}\} \subset \mathrm{Fix}\, S^{**}.$$

It is easy to see that $\tilde{U}$ as a subspace of U^{**} is infinite dimensional. Hence U^{**} contains a copy of c_0, [8]. Using local reflexivity, [6], we see that there is a constant $c > 0$ such that U contains $\tilde{E}_N$ with $\sup_N d(\tilde{E}_N, l_\infty^N) \leqslant c$.

Fix N and let $e_1, ..., e_N \in \tilde{E}_N$ correspond to the unit vector basis of l_∞^N, say $e_j = \{e_{j,n}\}_{n=1}^\infty \in U$ and $\max |\lambda_j| \leqslant \|\sum_{j=1}^N \lambda_j e_j\| \leqslant c \max |\lambda_j|$ for all λ_j. Now fix ε, $0<\varepsilon<1$, and let Ω be an ε/N-net of $\{\lambda \in \mathbf{C}: |\lambda| \leqslant 1\}$. Using the definition of ultraproduct we find n such that

$$(1-\varepsilon) \max |\lambda_j| \leqslant \left\| \sum_{j=1}^{N} \lambda_j e_{j,n} \right\| \leqslant (c+\varepsilon) \max |\lambda_j|$$

whenever $(\lambda_1, ..., \lambda_N) \in \Omega^N$. If ε is small enough this clearly implies that

$$E_n := \operatorname{span}\{e_{j,n}: j=1, ..., N\} \subset (R_n - R_{n-1}) X$$

is $2c$-isomorphic to l_∞^N. Now, find a projection $\tilde{Q}_N: X \to E_N$ with $\|\tilde{Q}_N\| \leqslant 2c$ and put $Q_N = \tilde{Q}_N(R_{n+1} - R_{n-2})$. Hence $R_j Q_N = Q_N R_j = 0$ for $j < n-2$ and $R_m Q_N = Q_N R_m = Q_N$ for $m > n+1$. Discard $n-2, ..., n+1$ and go over to a suitable subsequence of $\{R_n\}_{n=1}^\infty$ as before.

The Case $p=1$. This follows from the preceding case by duality in a similar fashion as the case $1<p\leqslant 2$ from the case $2 \leqslant p < \infty$. ∎

As an immediate consequence of Lemma 2.3 we obtain

COROLLARY 2.4. *Assume* $\sup_m \dim(R_m - R_{m-1}) X = \infty$. *Then there is a constant* $c>0$ *such that for each pair of positive integers* (m, N) *there is an integer* $n>m$, *a subspace* $E \subset (R_n - R_m) X$ *and a projection* $Q: X \to E$ *with* $\|Q\| \leqslant c$ *and* $d(E, l_p^N) \leqslant c$.

LEMMA 2.5. *Assume* $\sup_m \dim(R_m - R_{m-1}) X = \infty$. *Then there are integers* $0 \leqslant n_1 < n_2 < \cdots$ *and a c.a.s.* $\{P_n\}_{n=1}^\infty$ *of* X *such that* $P_n - P_{n-1}$ *factors uniformly through* l_p^m*'s and the* P_{n_k}, $k=1, 2, ...$, *are projections.*

Proof. We may assume that w.l.o.g. $\{R_n\}_{n=1}^\infty$ satisfies the assertions of Lemma 2.3., otherwise go over to a suitable subsequence $\{\hat{R}_n\}_{n=1}^\infty$. Go back to Y of (2.4) and i: $\cup \operatorname{Fix} R_n \to Y$. For each m find $Y_m \subset Y$ with $\sup_m d(Y_m, l_p^{\dim Y_m}) < \infty$ and

$$R_{m+1} Y \subset Y_m \tag{2.6}$$

in accordance with Lemma 2.1. We define $n_0, n_1, n_2, ...$ by induction. Start with $n_0 = 0 = n_1$, $P_0 = 0$. Then assume that we have already $n_0, ..., n_m$ and $P_{n_0}, ..., P_{n_{m-1}}$ with

$$P_{n_j} R_n = P_{n_j} = R_n P_{n_j} \qquad \text{for all} \quad n \geqslant n_m - 1 \qquad \text{and} \qquad j<m. \tag{2.7}$$

Find, in view of Lemma 2.3., $n_{m+1} > n_m + 2$ such that X contains a subspace $E \sim Y_{n_m}$ with a projection $Q_m \colon X \to E$ satisfying

$$Q_m R_k = R_k Q_m = \begin{cases} Q_m, & k \geqslant n_{m+1} - 1 \\ 0, & \text{else} \end{cases} \tag{2.8}$$

In particular, $E \subset \operatorname{Fix} R_{n_{m+1}-1} \cap \ker R_{n_{m+1}-2}$. Moreover, we can assume that $\|Q_m\|$ and $d(E, Y_{n_m})$ depend only on λ. Let $V_m \colon E \to Y_{n_m}$ be an isomorphism with $\|V_m\| \cdot \|V_m^{-1}\| = d(E, Y_{n_m})$. (Remember, E is finite dimensional.) Put

$$\begin{aligned} P_{n_m} x = {} & R_{n_m} x + i^{-1}(R_{n_m} - R_{n_m - 1}) V_m Q_m x \\ & + V_m^{-1}(id - R_{n_m})\, V_m Q_m x + V_m^{-1}\, i(R_{n_m+1} - R_{n_m})\, x, \; x \in X. \end{aligned} \tag{2.9}$$

The definition (2.9) makes sense if we take into account (2.6). Moreover $\|P_{n_m}\|$ depends only on λ. (2.8) proves $P_{n_m} R_n = R_n P_{n_m} = P_{n_m}$ for all $n \geqslant n_{m+1} - 1$. We have, in view of (2.6), (2.7), (2.8),

$$P_{n_j} R_{n_m} = P_{n_j} = R_{n_m} P_{n_j} \qquad \text{and} \qquad P_{n_j} Q_m = 0 = Q_m P_{n_j} \qquad \text{for} \quad j < m.$$

Hence we obtain $P_{n_m} P_{n_j} = P_{n_j} = P_{n_j} R_{n_m - 1} P_{n_m} = P_{n_j} P_{n_m}$ for $j < m$. Furthermore, an elementary calculation based on (2.8), together with (2.6), (2.7), shows that $P_{n_m}^2 = P_{n_m}$. (Note that $Q_m X = E = V_m^{-1}\, Y_{n_m} \subset \ker R_{n_m} \cap \ker R_{n_m+1} \cap \ker R_{n_m - 1}$ and $V_m Q_m V_m^{-1} = id$ on Y_{n_m}.) Hence $\{P_{n_m}\}_{m=1}^{\infty}$ defines an FDD on X. To complete the proof we fill in gaps. Take an integer j such that $j \neq n_k - 1, n_k, n_k + 1$ for all k, say $n_m + 1 < j < n_{m+1} - 1$. Put $P_j = R_j + Q_m$. Then, with (2.7), (2.8), we conclude that $P_h, h \neq n_k - 1, n_k + 1$ for all k, is a c.a.s. for X. (Note that (2.8) yields $Q_m Q_{m+1} = 0 = Q_{m+1} Q_m$.) It remains to show that the differences factor uniformly through l_p^m's. This is clear for $P_j - P_{j-1}$, if $n_m + 1 < j - 1 < j < n_{m+1} - 1$ for some m. If $j = n_m + 2$ then (2.9.) yields

$$\begin{aligned} P_{n_m+2} - P_{n_m} = {} & Q_m + (R_{n_m+2} - R_{n_m+1}) \\ & + (R_{n_m+1} - R_{n_m}) - i^{-1}(R_{n_m} - R_{n_m-1}) V_m Q_m \\ & - V_m^{-1}(id - R_{n_m}) V_m Q_m - V_m^{-1}\, i(R_{n_m+1} - R_{n_m}). \end{aligned}$$

Since Q_m is a projection onto an l_p^k-space, in view of (2.1), all six preceding summands factor uniformly through l_p^k's. Hence $P_{n_m+2} - P_{n_m}$ factors uniformly through l_p^k's. The proof is similar for $P_{n_m} - P_{n_m-2}$. ∎

3. Proof of Theorem I

The proof of Theorem I is split into two parts. At first we have

Proposition 3.1. *Let $\{R_n\}_{n=1}^{\infty}$ be a c.a.s. for the separable Banach space X such that $\sup_n \dim(R_n - R_{n-1}) X < \infty$. Then X has a basis.*

Proof. By assumption, since $R_n(id - R_n) X \subset (R_n - R_{n-1}) X$, we find uniformly bounded projections $Q_n : X \to R_n(id - R_n) X$ for all n. Put $P_n = R_n + Q_n(R_{n+1} - R_n)$. Clearly, in view of (1.2), $P_n P_m = P_{\min(n,m)}$ if $|n-m| \geqslant 2$. Moreover,

$$P_n^2 = R_n^2 + R_n Q_n(R_{n+1} - R_n) + Q_n R_n(R_{n+1} - R_n)$$
$$+ Q_n(R_{n+1} - R_n)\, Q_n(R_{n+1} - R_n) = P_n.$$

Hence $\{P_{2n}\}_{n=1}^{\infty}$ defines an FDD on X. We have

$$P_{2n} - P_{2n-2} = (R_{2n} - R_{2n-1}) + (R_{2n-1} - R_{2n-2})$$
$$+ Q_{2n}(R_{2n+1} - R_{2n}) - Q_{2n-2}(R_{2n-1} - R_{2n-2})$$

and $Q_m(R_{m+1} - R_m) X \subset (R_m - R_{m-1}) X$ for all m. This shows that, by assumption, $\sup_n \dim(P_{2n} - P_{2n-2}) X < \infty$. Then, clearly, X has a basis. ∎

For the second part we need another Lemma.

Lemma 3.2. *Let X have a c.a.s. $\{P_n\}_{n=1}^{\infty}$ such that all P_n are projections. Consider some p with $1 \leqslant p \leqslant \infty$. Assume that the following two conditions hold*:

(3.1) *There is a constant $\lambda \geqslant 1$ and, for every pair of integers (m, n) with $0 < m < n$, there is a λ-complemented subspace $F_{n,m}$ of some l_p^k-space such that $((P_n - P_m) X \oplus F_{n,m})_{(p)}$ has a basis whose basis constant depends only on λ.*

(3.2) *For every pair (k, m) of positive integers there is an integer $n > m$ such that $(P_n - P_m) X$ contains an isometric copy of l_p^k which is 1-complemented in $(P_n - P_m) X$.*

Then X has a basis.

Proof. We use induction to find $0 < n_1 < n_2 < \cdots$ and subspaces F_{m-1}, G_m satisfying the following: $(P_{n_m} - P_{n_{m-1}}) X = F_{m-1} \oplus G_m$, F_{m-1} is λ-complemented in an isometric copy of $l_p^{k_{m-1}}$, the norm of the projection onto F_{m-1} along G_m depends only on λ and $G_{m-1} \oplus F_{m-1}$ has a basis whose basis constant depends only on λ. Start with $n_1 = 1$, $G_1 = P_1 X$, $F_0 = \{0\}$. Then assume that we have already F_{m-1}, G_m, n_m. Using (3.1) find F which is λ-complemented in some l_p^k such that $((P_{n_m} - P_{n_{m-1}})\, X \oplus F)_{(p)}$ has a basis whose basis constant depends only on λ. Hence $(F_{m-1} \oplus F)_{(p)}$ is isometric to a λ-complemented subspace of a suitable $l_p^{k_m}$. Find, according to (3.2), an integer $n_{m+1} > n_m$ and an isometric copy V of $l_p^{k_m}$ which is 1-complemented in $(P_{n_{m+1}} - P_{n_m})\, X$. Let $F_m \subset V$ correspond to the subspace of $l_p^{k_m}$ which is isometric to $(F_{m-1} \oplus F)_{(p)}$. We have $G_m \subset (P_{n_m} - P_{n_{m-1}})\, X$ and $F_m \subset (P_{n_{m+1}} - P_{n_m})\, X$. Hence

$$G_m \oplus F_m \cong G_m \oplus (F_{m-1} \oplus F)_{(p)} \sim ((P_{n_m} - P_{n_{m-1}})\, X \oplus F)_{(p)}$$

which shows that $G_m \oplus F_m$ has a basis whose basis constant depends only on λ. By assumption on P_n we obtain

$$\begin{aligned} X &\sim P_{n_1} X \oplus (P_{n_2} - P_{n_1})\, X \oplus (P_{n_3} - P_{n_2})\, X \oplus \cdots \\ &= G_1 \oplus F_1 \oplus G_2 \oplus F_2 \oplus G_3 \oplus \cdots \end{aligned}$$

which shows that X has a basis. ∎

PROPOSITION 3.3. *Let $\{R_n\}_{n=1}^{\infty}$ be a c.a.s. of X such that $R_n - R_{n-1}$ factors uniformly through l_p^m's for some p and* $\sup_n \dim(R_n - R_{n-1})\, X = \infty$. *Then X has a basis.*

Proof. Using Lemma 2.5. we find a c.a.s. $\{P_n\}_{n=1}^{\infty}$ of X and integers $n_1 < n_2 < \ldots$ such that $P_n - P_{n-1}$ factors uniformly through l_p^m's and P_{n_k}, $k = 1, 2, \ldots$, are projections. By Lemma 2.2. applied to $\tilde{X} := (P_{n_k} - P_{n_h})\, X$ and $\{P_{j|\tilde{X}}\}_{j=1}^{\infty}$ there is a constant $c > 0$ satisfying the following: For each pair (k_1, k_2) of integers with $0 < k_1 < k_2$ there is a c-complemented subspace $F \subset l_p^M$ for some M such that $((P_{n_{k_2}} - P_{n_{k_1}})\, X \oplus F)_{(p)}$ has a basis with constant $\leqslant c$. With Corollary 2.4. we find an increasing sequence of integers $m(k)$, subspaces $E_k \subset (P_{n_{m(k+1)}} - P_{n_{m(k)}})\, X$ and projections Q_k: $X \to E_k$ with $d(E_k, l_p^k) \leqslant c$ and $\|Q_k\| \leqslant c$. Renorm X as follows: Let L_k: $E_k \to l_p^k$ be an isomorphism with $\|x\| \leqslant \|L_k x\| \leqslant c\, \|x\|$ for $x \in E_k$. Put, for $x \in X$, $|||x||| = \max(\|x\|, \sup_k \|L_k Q_k x\|)$. Then, after renorming, $E_k \cong l_p^k$ and E_k is 1-complemented in X. Now, (3.1) and (3.2) are satisfied for $\{P_{n_k}\}_{k=1}^{\infty}$. Lemma 3.2. completes the proof of Proposition 3.3. ∎

Of course, Propositions 3.1. and 3.3. prove Theorem I.

4. C_Λ- AND L_Λ-SPACES

Let $\|\cdot\|$ be the sup-norm or the $L_1(\mathbf{T})$-norm on $\mathbf{T} = \{z\colon |z| = 1\}$. We deal with the following classical operators (defined on $L_p(\mathbf{T})$, $p \in \{1, \infty\}$)

$$(\sigma_n f)(z) = \sum_{k=-n}^{n} \frac{n-|k|}{n} \hat{f}(k) z^k,$$

$$\begin{aligned}(V_{m,n} f)(z) &= \frac{n}{n-m} (\sigma_n f)(z) - \frac{m}{n-m} (\sigma_m f)(z) \\ &= \sum_{k=-m}^{m} \hat{f}(k) z^k + \sum_{m<|k|\leqslant n} \frac{n-|k|}{n-m} \hat{f}(k) z^k,\end{aligned} \tag{4.1}$$

$$(Rf)(z) = \sum_{k \geqslant 0} \hat{f}(k)\, z^k \text{ (here } f \text{ is a trigonometric polynomial)}$$

It is well known, [3], that

$$\|\sigma_n\| = 1 \quad \text{and} \quad \|V_{m,n}\| \leqslant \frac{n+m}{n-m} \quad \text{for all } n \quad \text{and} \quad m < n. \tag{4.2}$$

LEMMA 4.1. *There is a constant $c > 0$ such that*

$$\|R(V_{n_2,n_3} - V_{n_1,n_2})\| \leqslant c \left(1 + \log\left(\frac{n_3}{n_1}\right)\right) \|(V_{n_2,n_3} - V_{n_1,n_2})\|$$

whenever $1 \leqslant n_1 < n_2 < n_3$.

Proof. Let p and q be integers such that $2^p \leqslant n_1 < 2^{p+1}$ and $2^{q-1} < n_3 \leqslant 2^q$. Then $q - p \leqslant c_1(1 + \log(n_3/n_1))$ for some constant c_1. We have, by (4.1),

$$(R(V_{2^j,2^{j+1}} - V_{2^{j-1},2^j}) f)(z) = z^{2^j} \sigma_{2^j}(\bar{z}^{2^j} f(z)) - \tfrac{1}{2} z^{2^{j-1}} \sigma_{2^{j-1}}(\bar{z}^{2^{j-1}} f(z))$$

and

$$\begin{aligned} R(V_{n_2,n_3} - V_{n_1,n_2}) &= R(V_{2^q,2^{q+1}} - V_{2^{p-1},2^p})(V_{n_2,n_3} - V_{n_1,n_2}) \\ &= \sum_{j=p}^{q} R(V_{2^j,2^{j+1}} - V_{2^{j-1},2^j})(V_{n_2,n_3} - V_{n_1,n_2}) \end{aligned}$$

(Notice that the image of $V_{n_2,n_3} - V_{n_1,n_2}$ is spanned by z^k, $n_1 < |k| < n_3$, which are fix points for $V_{2^q,2^{q+1}} - V_{2^{p-1},2^p}$.) Hence (4.2) yields

$$\|R(V_{n_2,n_3} - V_{n_1,n_2})\| \leqslant c \left(1 + \log\left(\frac{n_3}{n_1}\right)\right) \|(V_{n_2,n_3} - V_{n_1,n_2})\|. \quad \blacksquare$$

For a subset $\Lambda \subset \mathbf{Z}$ let P_Λ be the projection with

$$P_\Lambda z^k = \begin{cases} z^k & k \in \Lambda \\ 0, & \text{else} \end{cases} \tag{4.3}$$

For special Λ this definition yields well defined bounded projections on $C(\mathbf{T})$ as well as on $L_1(\mathbf{T})$.

LEMMA 4.2. *Let $\Lambda = \bigcup_{k=1}^{r}(m_k \mathbf{Z} + q_k)$ for some integers m_k and q_k and fix $p \in \{1, \infty\}$.*

(a) *Then $\|P_\Lambda\| \leqslant r2^r$.*

(b) *Let m, n be integers with $0 < m < n$, put $\Omega = \Lambda \cap [m, n]$ and $A = \operatorname{span}\{z^k : k \in \Omega\}$. Then there is a constant λ, independent of m and n, and a c.a.s. $R_0, ..., R_q$ of A such that $R_0 = 0$, $R_q = id$, $R_j - R_{j-1}$ factor uniformly through l_p^m's with respect to λ and $\|R_j\| \leqslant \lambda$ for all j.*

Proof. (a) If $m_k \neq 0$ we can assume $m_k \geqslant 1$ and $0 \leqslant q_k \leqslant m_k - 1$. Then put

$$(Q_k f)(z) = \frac{1}{m_k} \sum_{j=0}^{m_k - 1} \exp\left(-\frac{2\pi i}{m_k} j q_k\right) f\left(\exp\left(\frac{2\pi i}{m_k} j\right) z\right)$$

If $m_k = 0$ put $(Q_k f)(z) = \hat{f}(q_k)\, z^{q_k}$. In any case, Q_k is contractive and $Q_k = P_{m_k \mathbf{Z} + q_k}$. We have

$$P_\Lambda = Q_1 + \sum_{h=1}^{r-1} \prod_{j=1}^{h} (id - Q_j)\, Q_{h+1}$$

which proves (a).

(b) We can assume that $n - m \geqslant 4$ and that $n - m$ is even (otherwise take $n - 1$ instead of n). Let q be the integer with $2^q \leqslant (n-m)/2 < 2^{q+1}$ and put $n_k = 2^k$, $k = 1, ..., q-1$, $n_q = (n-m)/2$. Hence $n_1 = 2 < n_2 < \cdots < n_q$ and

$$\frac{n_{k+1} + n_k}{n_{k+1} - n_k} \leqslant 8 \qquad \text{and} \qquad \frac{n_{k+1}}{n_{k-1}} \leqslant 8 \qquad \text{for all} \quad k. \tag{4.4}$$

Define $R_j : A \to A$ by $R_0 = 0$, $R_q = id$ and, for $1 \leqslant j \leqslant q-1$,

$$R_{q-j} z^k = z^n (id - V_{n_j, n_{j+1}})\, z^{m-n} (id - V_{n_j, n_{j+1}})\, z^{k-m}$$

$$= \begin{cases} z^k, & m + n_{j+1} \leqslant k \leqslant n - n_{j+1} \\ \dfrac{k - m - n_j}{n_{j+1} - n_j} z^k, & m + n_j \leqslant k < m + n_{j+1} \\ \dfrac{n - n_j - k}{n_{j+1} - n_j} z^k, & n - n_{j+1} < k \leqslant n - n_j \\ 0, & m \leqslant k \leqslant m + n_j \quad \text{or} \quad n - n_j \leqslant k \leqslant n \end{cases}$$

(4.2) and (4.4) yield $\|R_j\| \leqslant (1+8)^2$ for all j. By definition, $\{R_j\}_{j=1}^{\infty}$ is a c.a.s. for A. For $1 \leqslant j \leqslant q-1$ and $g \in L_p(\mathbf{T})$ put

$$(\tilde{S}_j g)(z) = z^m R(V_{n_j, n_{j+1}} - V_{n_{j-1}, n_j})\, \bar{z}^m g(z) \\ + z^n (id - R)(V_{n_j, n_{j+1}} - V_{n_{j-1}, n_j})\, \bar{z}^n g(z)$$

By (4.4) and Lemma 4.1. the $\tilde{S}_j$ are uniformly bounded. Define S_j: $L_p(\mathbf{T}) \to A$ by $S_j = P_A \tilde{S}_j$ and T_j: $A \to L_p(\mathbf{T})$ by $T_j = id$. Then $R_{q-j+1} - R_{q-j} = S_j T_j$ for $1 < j < q$. Hence $R_{q-j+1} - R_{q-j}$ factors uniformly through $L_p(\mathbf{T})$ and therefore also through suitable l_p^m's. Furthermore, $m := \dim(R_q - R_{q-1})A \leqslant 4$ since $n_1 = 2, n_2 \leqslant 4$. Hence $d((R_q - R_{q-1})A), l_p^m) \leqslant 4$. Finally, in view of $n_q = (n-m)/2$,

$$R_1 z^k = (R_1 - R_0)\, z^k$$

$$= \begin{cases} \dfrac{k-m-n_{q-1}}{n_q - n_{q-1}}\, z^k, & m + n_{q-1} \leqslant k < (n+m)/2 \\ \dfrac{n-k-n_{q-1}}{n_q - n_{q-1}}\, z^k, & (n+m)/2 < k \leqslant n - n_{q-1} \\ 0, & \text{otherwise} \end{cases}$$

Put $T_q = id$ and define S_q: $L_p(\mathbf{T}) \to A$ by

$$(S_q g)(z) = P_A z^{(n+m)/2} \sigma_{n_q - n_{q-1}} \bar{z}^{(n+m)/2} f(z).$$

We obtain $\|S_q\| \leqslant \|P_A\|$ and $S_q T_q = R_1 = R_1 - R_0$. This concludes the proof. ∎

Proof of Theorem II. We treat the C_A-case. The proof for the L_A-case is identical. Consider a sequence $\{n_k\}_{k=0}^{\infty}$ satisfying (1.3). Find an increasing sequence of integers m_j such that

$$m_{3k} = n_k, \quad k = 1, 2, ..., \tag{4.5}$$

and there are constants $c, d > 0$ with

$$cm_j \leqslant m_{j+1} - m_j \leqslant dm_j \qquad \text{for all} \quad j = 1, 2, ... \tag{4.6}$$

Put $m_{-j} = -m_j$, $j = 1, 2, ...$ Define $R_j = V_{m_j, m_{j+1}}$, $j = 1, 2, ...$ By (4.2), (4.6), $\{R_j\}_{j=1}^{\infty}$ is uniformly bounded on C_A. In view of (4.1), $R_{j|C_A}$ is a c.a.s. for C_A. We show that $(R_{j+1} - R_j)_{|CA}$ factors uniformly through l_∞^m's. Then Theorem II follows from Theorem I. By (4.6) and Lemma 4.1., $R(R_{j+1} - R_j)$

and $(id-R)(R_{j+1}-R_j)$ are uniformly bounded. Moreover, by (4.5), for every j, there is k with $[m_{j-1}, m_{j+3}] \subset [n_k, n_{k+2}]$. Hence

$$R(R_{j+2}-R_{j-1})\,C_\Lambda \subset \operatorname{span}\{z^h\colon h\in[n_k, n_{k+2}]\cap\Lambda\}.$$

By assumption there is a subspace $Y_j \subset C(\mathbf{T})$ with

$$R(R_{j+2}-R_{j-1})\,C_\Lambda \subset Y_j,\ R(R_{j+1}-R_j)\,Y_j \subset C_\Lambda \quad\text{and}\quad d(Y_j, l_\infty^{\dim Y_j}) \leqslant \lambda.$$

Define $T_j\colon C_\Lambda \to Y_j$ by $T_j f = R(R_{j+2}-R_{j-1})\,f$ and $S_j\colon Y_j \to C_\Lambda$ by $S_j g = R(R_{j+1}-R_j)\,g$. Then $S_j T_j = R(R_{j+1}-R_j)$. Hence $R(R_{j+1}-R_j)$ factors uniformly through l_∞^m's. One shows similarly that $(id-R)(R_{j+1}-R_j)$ factors uniformly through l_∞^m's (consider the negative integers). Thus $R_{j+1}-R_j$ factors uniformly through l_∞^m's and the proof is completed. ∎

Proof of the Corollary. Put $\Omega_k = \bigcup_{j=1}^q Z_{k,j}$. By Lemma 4.2.(a), C_{Ω_k} and L_{Ω_k} are uniformly complemented in $C(\mathbf{T})$ and $L_1(\mathbf{T})$, resp. Hence, by [5], there is some $\lambda > 1$, independent of k, such that for any finite dimensional subspace $E \subset C_{\Omega_k}$ (resp. $E \subset L_{\Omega_k}$) there exists another finite dimensional subspace $F \subset C_{\Omega_k}$ (resp. $F \subset L_{\Omega_k}$) with $E \subset F$ and $d(F, l_\infty^{\dim F}) < \lambda$ (resp. $d(F, l_1^{\dim F}) < \lambda$). Thus the corollary follows from Theorem II. ∎

Proof of Theorem III. We prove the C_Λ-case. The proof of the L_Λ-case is identical. Consider a sequence $\{n_k\}_{k=0}^\infty$ with (1.3) and let $F_j\colon C_\Lambda \to C_\Lambda$ be the projection with

$$F_j z^k = \begin{cases} z^k, & |k| \leqslant n_j \\ 0, & \text{else} \end{cases}, \quad j = 1, 2, ...$$

We claim, $\{F_j\}_{j=1}^\infty$ is uniformly bounded. Indeed, put $\Omega_j = \bigcup_{h=1}^q Z_{j,h}$ where $\Lambda \cap [n_j, n_{j+1}] = \Omega_j \cap [n_j, n_{j+1}]$. For $f \in C_\Lambda$, with (4.3), we have

$$\begin{aligned}(F_j f)(z) = (V_{n_j, n_{j+1}} f)(z) - P_{\Omega_j} z^{n_j} (\sigma_{n_{j+1}-n_j} \bar{z}^{n_j} f)(z)\\ - p_{\Omega_{-j-1}} \bar{z}^{n_j} (\sigma_{n_{j+1}-n_j} z^{n_j} f)(z)\end{aligned}$$

This is a consequence of the assumption $\Omega_j \cap \Omega_{j-1} = \varnothing = \Omega_{-j-1} \cap \Omega_{-j}$, that is

$$P_{\Omega_j} z^{n_j} \sigma_{n_{j+1}-n_j} \bar{z}^{n_j} z^h = 0, \quad h \in [n_{j-1}, n_j] \cap \Lambda,$$

and

$$P_{\Omega_{j-1}} \bar{z}^{n_j} \sigma_{n_{j+1}-n_j} z^{n_j} z^h = 0, \quad h \in [n_{j-1}, n_{-j+1}] \cap \Lambda.$$

Moreover, by (1.3), $\|V_{n_j, n_{j+1}}\| \leqslant (b+2)/a$. Hence with Lemma 4.2.(a), $\{F_j\}_{j=1}^{\infty}$ is uniformly bounded. Lemma 4.2.(b) yields a c.a.s. $R_{j,k}^{+}$, $k = 0, 1, ..., u_j$, for $C_{\Omega_j \cap [n_j, n_{j+1}]}$ such that $R_{j,0}^{+} = 0$, $R_{j,u_j}^{+} = id$ and $R_{j,k+1}^{+} - R_{j,k}^{+}$ factors uniformly through l_{∞}^{m}'s. Furthermore find a c.a.s. $R_{j,k}^{-}$, $k = 0, 1, ..., v_j$, on $C_{\Omega_{j-1} \cap [n_{-j-1}, n_{-j}]}$ with the corresponding properties. We may assume $u_j = v_j$ (otherwise count R_{j,u_j}^{+} or R_{j,v_j}^{-} several times). Now define

$$R_{j,k} = F_j + R_{j,k}^{+} R(F_{j+1} - F_j) + R_{j,k}^{-}(id - R)(F_{j+1} - F_j), \quad k = 0, 1, ..., u_j.$$

Consider the lexicographical order for the indices. We claim that $\{R_{j,k}\}$ is uniformly bounded. Then $\{R_{j,k}\}$ is a c.a.s. To this end we observe that, on C_{Λ}

$$R(F_{j+1} - F_j) = P_{\Omega_j}(R(V_{n_{j+1}, n_{j+2}} - V_{n_j, n_{j+1}}) + R(V_{n_j, n_{j+1}} - V_{n_{j-1}, n_j}))$$

Since $n_{j+2}/n_j, n_{j+1}/n_{j-1} \leqslant (b+1)^2$, Lemma 4.2. implies that $\{R(F_{j+1} - F_j)\}_{j=1}^{\infty}$ is uniformly bounded. Hence $\{(id - R)(F_{j+1} - F_j)\}_{j=1}^{\infty}$ is uniformly bounded. This proves the claim. By definition of $R_{j,k}$ we see that the differences factor uniformly through l_{∞}^{m}'s. Now, Theorem I concludes the proof. ∎

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